

REGULARITY OF WEAK SOLUTIONS TO RATE-INDEPENDENT SYSTEMS IN ONE-DIMENSION

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ABSTRACT. We show that under some appropriate assumptions, every *weak solution* (e.g. energetic solution) to a given rate-independent system is of class SBV, or has finite jumps, or is even piecewise C^1 . Our assumption is essentially imposed on the energy functional, but not convexity is required.

1. INTRODUCTION

The rate independency is the property indicating to those systems which are subjected by the external loading on the time scale that is much slower than any internal time scale, but still much faster than the time so that the system reaches equilibrium, so that the inertia and kinetic energies can be neglected. The main feature of rate-independent systems is that the changes of the rate of the solutions essentially depends on the changes of the velocity of the loading, namely if the loading acts twice faster, then the solutions also respond twice faster. Rate-independent systems are used to characterize many physical phenomena involved in plasticity, phase transformation (electromagnetism, superconductivity or dry friction on surfaces), and some certain hysteresis models (shape-memory alloys, quasistatic delamination, fracture, etc.). For a detailed discussion on the rate-independent systems, we refer to the books [4, 14, 19, 2].

In this paper, we are interested in the regularity of weak solutions to one-dimensional rate-independent systems. In one-dimension, a rate-independent system is characterized by an energy functional $\mathcal{E} \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$ and a dissipation function, which we will take the usual distance $|\cdot|$ for simplicity. A BV function $x : [0, T] \rightarrow \mathbb{R}$ is called a *weak solution* to the rate-independent system with the initial position $x_0 \in \mathbb{R}$ if $x(0) = x_0$, and $x(\cdot)$ satisfies

(i) the *weak local stability*, that

$$|\partial_x \mathcal{E}(t, x(t))| \leq 1 \quad (1)$$

for every $t \in [0, T]$ such that $x(\cdot)$ is continuous at t , and

(ii) the *energy-dissipation upper bound*, that

$$\mathcal{E}(t_2, x(t_2)) - \mathcal{E}(t_1, x(t_1)) \leq \int_{t_1}^{t_2} \partial_t \mathcal{E}(s, x(s)) ds - \mathcal{Diss}(x(\cdot); [t_1, t_2]), \quad (2)$$

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for all $0 \leq t_1 \leq t_2 \leq T$.

Here we define the dissipation energy

$$\mathcal{Diss}(x(\cdot); [t_1, t_2]) := \sup \left\{ \sum_{n=1}^N |x(s_n) - x(s_{n-1})| \mid N \in \mathbb{N}, t_1 \leq s_0 < s_1 < \dots < s_N \leq t_2 \right\}.$$

A particular case of weak solutions is the *energetic solutions*, which was first introduced by Mielke and Theil [11] and further studied in [12, 6, 3, 7]. A function $u : [0, T] \rightarrow \mathbb{R}$ is called an energetic solution to the rate-independent system with the initial position $x_0 \in \mathbb{R}$ if $x(0) = x_0$ and $x(\cdot)$ satisfies

(i) the *global stability*, that

$$\mathcal{E}(t, u(t)) \leq \mathcal{E}(t, x) + |x - u(t)| \quad (3)$$

for all $(t, x) \in [0, T] \times \mathbb{R}$, and

(ii) and the *energy-dissipation balance*, that

$$\mathcal{E}(t_2, u(t_2)) - \mathcal{E}(t_1, u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathcal{E}(s, u(s)) ds - \mathcal{Diss}(u; [t_1, t_2]). \quad (4)$$

for all $0 \leq t_1 < t_2 \leq T$.

However, our notion of weak solutions also contains BV solutions [10], local solutions [18], parametrized solutions [9] and epsilon-stable solutions [5].

When the energy functional is *convex*, the regularity was already investigated by Mielke, Rossi and Thomas [8, 17]. They showed that if the energy functional $\mathcal{E}(t, \cdot)$ is α -convex, $\partial_t \mathcal{E}(t, \cdot)$ is Lipschitz continuous (or Hölder continuous), and

$$\partial_t \mathcal{E}(t, x) \leq \lambda |\mathcal{E}(t, x)| \quad \forall t \in [0, T]$$

for some constant $\lambda > 0$, then every energetic solution is Lipschitz continuous (or Hölder continuous, respectively). Moreover, if the energy functional has the form $\mathcal{E}(t, x) = W(x) - \ell(t)x$, where $W(x)$ is the double-well potential and $\ell(t)$ is a smooth loading, Stefanelli [16] proposed a variational characterization of rate-independent evolution. Later, if $\mathcal{E}(t, x) = W(x) - \ell(t)x$ for a general smooth potential $W(x)$ and a monotone loading function $\ell(t)$, Rossi and Savaré [15] derived a full characterization of all energetic and BV solutions to rate-independent systems in one-dimension.

However, in the general case (in particular when the energy functional is *non-convex*), the solutions may behave badly, as we can see in the following

Theorem 1 (Any increasing function is an energetic solution). *Let $u : [0, T] \rightarrow \mathbb{R}$ be an arbitrary increasing and left-continuous function. Then u is an energetic solution of some rate-independent system with smooth energy functional.*

In this paper, we shall prove that under some certain requirements (but not convexity) on the energy functional, any weak solution are of class SBV. Moreover, we give sufficient condition ensuring that every weak solution has only finitely many jumps, and it is piecewise C^1 -smooth. Our techniques, however, seem rather specific for one dimension, and we hope to come back to the regularity in higher dimensions in a future work.

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2. MAIN RESULTS

Our first regularity result deal with the SBV property of weak solutions. We shall need a technical condition.

(H1) $\mathcal{E}(t, x)$ is of class C^3 and the set

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xxx} \mathcal{E}(t, x) = 0\}$$

has only finitely many elements.

Note that no convexity is imposed. We have

Theorem 2 (SBV regularity). *Assume that (H1) holds true. Then every BV function $x(\cdot)$ satisfying the weak local stability (1) and the energy-dissipation upper bound (2) must be of class SBV.*

Remark. The SBV regularity still holds if the set in (H1) is at most countable (instead of finite). Moreover, the result can be generalized in higher dimensions as follows (see [13] for a detailed proof). We assume that the energy functional $\mathcal{E}(t, x)$ is of class C^3 and the set

$$\{(t, x) \in (0, T) \times \mathbb{R}^d \mid |\nabla_x \mathcal{E}(t, x)| = 1, G(t, x) = (\nabla_x \mathcal{E}(t, x)) \cdot (\nabla_x F(t, x)) = 0\}$$

is at most countable, where the function $F(t, x)$ is defined by

$$F(t, x) := (\nabla_x \mathcal{E}(t, x)) \cdot H(t, x) \cdot (\nabla_x \mathcal{E}(t, x))^T$$

with the Hessian matrix

$$[H(t, x)]_{ij} := (\partial_{x_i} \partial_{x_j} \mathcal{E})(t, x).$$

Then every BV function $x : [0, T] \rightarrow \mathbb{R}^d$ (with $d \geq 1$) satisfying the weak local stability (1) and the energy-dissipation upper bound (2) must be of class SBV.

In the next result, we consider the differentiability of weak solutions. By a technical reason, we have to replace the above weak local stability by the strong-local stability:

$$z = x(t) \text{ is a local minimizer of the functional } z \mapsto \mathcal{E}(t, z) + |z - x(t)| \quad (5)$$

for every $t \in [0, T] \setminus J$, where J is the jump set of $x(\cdot)$, which will be assumed to be finite. Moreover, we shall replace the condition (H1) on the energy functional by some of the followings.

(H2) The set

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xt} \mathcal{E}(t, x) = 0, \\ [\partial_{xxt} \mathcal{E}(t, x)]^2 = \partial_{xtt} \mathcal{E}(t, x) \cdot \partial_{xxx} \mathcal{E}(t, x)\}$$

has only finitely many elements.

(H3) The set

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xt} \mathcal{E}(t, x) = 0\}$$

has only finitely many elements.

(H4) The set

$$\{(t, x) \in [0, T] \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xt} \mathcal{E}(t, x) = \partial_{xtt} \mathcal{E}(t, x) = 0\}$$

is empty.

Theorem 3 (Differentiability). *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ has only finitely many jump points and satisfies the strong-local stability (5) and the energy-dissipation upper bound (2). Then we have the following statements.*

- (i) If (H1) holds true, then we can decompose $[0, T]$ into four disjoint sets I_1, I_2, I_3 and J such that

- For every $t \in I_1$, $x'(t)$ does not exist and either $x'_-(t) = 0$ or $x'_+(t) = 0$.
- For every $t \in I_2$, $x'_-(t)$ and $x'_+(t)$ do exist, but they are different. Moreover, $x(\cdot)$ is differentiable in a neighborhood of t (except the point t itself) and

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

- For every $t \in I_3$, $x(\cdot)$ is differentiable at t , namely $x'(t)$ exists.
- J is the jump set of $x(\cdot)$.

Notice that both I_1 and I_2 are discrete sets. Moreover, if (H1) and (H2) holds true, then $I_1 \cup I_2$ is also a discrete set.

- (ii) If (H1) and (H3) hold true, then there exists a set I of isolated points such that for any $t \in (0, T) \setminus I$, the (classical) derivative $x'(t)$ exists. Moreover, the function $x'(\cdot)$ is continuous on $(0, T) \setminus I$.
- (iii) If (H1), (H3) and (H4) hold true, then there exist finite disjoint open intervals $\{I_n\}_{n=1}^M$ such that $[0, T] = \cup_{n=1}^M \overline{I_n}$, and $x(\cdot)$ is C^1 on any interval I_n .

Here the right and left derivatives $x'_+(t)$, $x'_-(t)$ are defined by

$$x'_+(t) := \lim_{s \downarrow t} \frac{x(s) - x(t)}{s - t}, \quad x'_-(t) := \lim_{s \uparrow t} \frac{x(s) - x(t)}{s - t}.$$

Moreover, we say that s is an isolated point of I if there exists $\varepsilon > 0$ such that

$$(s - \varepsilon, s + \varepsilon) \cap I = \{s\}.$$

In Theorem 3, we have required, as a-priori, that the solution has finitely many jump points. In the last result, we give a sufficient condition on the energy functional to remove this assumption.

(H5) The set

$$\{(t, x) \in [0, T] \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = 0\}$$

is empty.

Theorem 4 (Finite jumps). *Assume that (H5) holds true. Then every BV function $x : [0, T] \rightarrow \mathbb{R}$ satisfying the weak local stability (1) and the energy-dissipation upper bound (2) must have only finitely many jumps.*

The proofs of the previous theorems are provided in the next sections.

3. PROOF OF THEOREM 1

We start by recalling a classical result.

Lemma 5. *If $u : [0, T] \mapsto \mathbb{R}$ is an increasing function, then there exists a smooth function $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} g(t, x) &\in [-1, 0] \text{ for all } t \in [0, T] \text{ and for all } x < u(t), \\ g(t, x) &\in (0, 1] \text{ for all } t \in [0, T] \text{ and for all } x > u(t), \\ g(t, x) &= 0 \text{ for all } t \in [0, T] \text{ and for all } x = u(t). \end{aligned}$$

The proof of Lemma 5 can be found in the Appendix. Now we give

Proof of Theorem 1. We choose the energy functional $\mathcal{E}(t, x)$ such that $g(t, x) = \partial_x \mathcal{E}(t, x) + 1$ (here g is from Lemma 5), the dissipation function $\Psi(x) := |x|$, and the initial value $x_0 := u(0)$. We shall prove that u is an energetic solution of the system $(\mathcal{E}, |\cdot|, u(0))$.

It is known that $x(\cdot)$ is an energetic solution to the system $(\mathcal{E}, |\cdot|, x_0)$ if the following three conditions hold (see Proposition 5.13 [1], or a simplified version in Proposition 1.4 [13]).

- (i) $x(\cdot)$ is left-continuous.
- (ii) $\mathcal{D}iss(x(\cdot); [0, T]) = |x(T) - x_0|$.
- (iii) For all $t \in [0, T]$, $x(t)$ minimizes the functional $x \mapsto \mathcal{E}(t, x) + |x - x_0|$ for $x \in \mathbb{R}$.

Thus it remains to check that u satisfies the condition (iii). We shall use the fact that for all t , $\partial_x \mathcal{E}(t, u(t)) = -1$ and $\partial_x \mathcal{E}(t, x) \in (-1, 0]$ if $x > u(t)$, and $\partial_x \mathcal{E}(t, x) \in [-2, -1]$ if $x < u(t)$. We distinguish two cases.

Case 1: $x > u(t)$. By the smoothness of \mathcal{E} , we can write

$$\begin{aligned} \mathcal{E}(t, x) &= \mathcal{E}(t, u(t)) + \int_{u(t)}^x \partial_x \mathcal{E}(t, z) dz \\ &> \mathcal{E}(t, u(t)) + \int_{u(t)}^x (-1) = \mathcal{E}(t, u(t)) + u(t) - x. \end{aligned}$$

Case 2: $x < u(t)$. Similarly to Case 1, we write

$$\begin{aligned} \mathcal{E}(t, x) &= \mathcal{E}(t, u(t)) - \int_x^{u(t)} \partial_x \mathcal{E}(t, z) dz \\ &> \mathcal{E}(t, u(t)) - \int_x^{u(t)} (-1) = \mathcal{E}(t, u(t)) + u(t) - x. \end{aligned}$$

Thus in both cases, we have

$$\begin{aligned} \mathcal{E}(t, x) + |x - u(0)| &> [\mathcal{E}(t, u(t)) + u(t) - x] + [x - u(0)] \\ &= \mathcal{E}(t, u(t)) + u(t) - u(0). \end{aligned}$$

In summary, $u(t)$ is the unique minimizer for the functional $x \mapsto \mathcal{E}(t, x) + |x - x(0)|$ over $x \in \mathbb{R}$. This completes the proof. \square

4. PROOF OF THEOREM 2

In this section, we prove Theorem 2.

Proof. Step 1. Thanks to Proposition 1.5 [13], we can assume that $x(\cdot)$ is right-continuous. By dividing $(0, T)$ into smaller intervals if necessary, we can assume that the set

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xxx} \mathcal{E}(t, x) = 0\}$$

is empty.

Step 2. Since $x(\cdot)$ is a BV function in 1-dim which is right-continuous, there is a real-valued Radon measure μ such that

$$x(t) = \text{const} + \mu((0, t]) \text{ for all } t \in [0, T].$$

By Lebesgue Decomposition Theorem we can write

$$\mu = f dx + \mu_s$$

where $f \in L^1$ and $\mu_s = \mu|_S$ with

$$S = \left\{ t \in (0, T) \mid \lim_{h \downarrow 0} \frac{|\mu|(t-h, t+h)}{h} = \infty \right\}.$$

Let J be the jump set of $x(\cdot)$. We split μ_s into the Cantor part $\mu_c := \mu|_{S \setminus J}$ and the jump part $\mu_J := \mu|_J$. To show that $x(\cdot)$ is of SBV , we need to prove that $\mu_c = 0$.

Step 3. Next, we shall use the following lemmas, which will be proved later.

Lemma 6. *For any BV function $x : [0, T] \rightarrow \mathbb{R}$ which is right-continuous, the set*

$$A := \left\{ t \in (0, T) \setminus J \mid \liminf_{h \rightarrow 0} \left| \frac{x(t+h) - x(t)}{h} \right| < \infty \right\}$$

has $|\mu_s|$ -measure 0.

Lemma 7. *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ satisfies the weak local stability (1) and the energy-dissipation upper bound (2). If (H1) holds true, then the set*

$$B := \left\{ t \in (0, T) \setminus J \mid \lim_{h \rightarrow 0} \left| \frac{x(t+h) - x(t)}{h} \right| = \infty \right\}$$

is at most countable. Therefore, $|\mu_s|(B) = 0$.

Step 4. Since μ_c is the restriction of μ_s on $(0, T) \setminus J$, $\mu_c = 0$ if $|\mu_s|((0, T) \setminus J) = 0$. Notice that $(0, T) \setminus J = A \cup B$. Hence, lemmas 6 and 7 ensure that $|\mu_s|((0, T) \setminus J) = 0$. This completes the proof of Theorem 2. \square

It remains to verify Lemma 6 and Lemma 7. Lemma 6 is a general fact of BV functions, and its proof can be found in the Appendix. On the other hand, the proof of Lemma 7 is based on the following observation, which is a key property of weak solutions to rate-independent systems.

Lemma 8. *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ satisfies the weak local stability (1) and the energy-dissipation upper bound (2). Then we have*

$$\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$$

for all $t \notin J \cup \text{int}(N \cup J)$. Here we denote by J the jump set of $x(\cdot)$ and $N := \{t \in (0, T) \mid x'(t) = 0\}$ is the null set of the derivative of $x(\cdot)$.

Proof. Step 1. First, we show that if $t \notin N \cup J$, then $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$.

Since $t \notin N$, we can find a sequence $t_n \rightarrow t$ and $t_n \neq t$ such that

$$\liminf_{n \rightarrow \infty} \left| \frac{x(t_n) - x(t)}{t_n - t} \right| > 0. \tag{6}$$

Case 1. Assume that $t_n \downarrow t$. From the energy-dissipation upper bound, one has

$$\mathcal{E}(t_n, x(t_n)) - \mathcal{E}(t, x(t)) \leq \int_t^{t_n} \partial_t \mathcal{E}(s, x(s)) ds - \mathcal{Diss}(x(\cdot); [t, t_n]).$$

Using Taylor's expansion on the left-hand side and the continuity of $s \mapsto \partial_t \mathcal{E}(s, x(s))$ on the right-hand side, we obtain

$$\begin{aligned} & \partial_t \mathcal{E}(t, x(t)) \cdot (t_n - t) + \partial_x \mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(x(t_n) - x(t)) + o(t_n - t) \\ & \leq (t_n - t) \cdot \partial_t \mathcal{E}(t, x(t)) - \mathcal{Diss}(x(\cdot); [t, t_n]) + o(t_n - t). \end{aligned}$$

Dividing this inequality by $|x(t_n) - x(t)|$ and using (6), we obtain

$$\partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \leq -\frac{\mathcal{Diss}(x(\cdot); [t, t_n])}{|x(t_n) - x(t)|} + o(1) \leq -1 + o(1). \quad (7)$$

Consequently, $|\partial_x \mathcal{E}(t, x(t))| \geq 1$. On the other hand, $|\partial_x \mathcal{E}(t, x(t))| \leq 1$ by the weak local stability (1). Thus $|\partial_x \mathcal{E}(t, x(t))| = 1$.

Case 2. Assume that $t_n \uparrow t$. From the energy-dissipation upper bound, one has

$$\mathcal{E}(t_n, x(t_n)) - \mathcal{E}(t, x(t)) \geq \int_t^{t_n} \partial_t \mathcal{E}(s, x(s)) ds + \mathcal{Diss}(x(\cdot); [t_n, t]).$$

Following the above proof, we obtain

$$\partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} \geq \frac{\mathcal{Diss}(x(\cdot); [t_n, t])}{|x(t_n) - x(t)|} + o(1) \geq 1 + o(1).$$

This also implies that $|\partial_x \mathcal{E}(t, x(t))| = 1$.

Step 2. We show that if $t \notin J$ and $t \notin \text{int}(N \cup J)$, then $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$.

Since $t \notin \text{int}(N \cup J)$, there exists a sequence $t_n \rightarrow t$ such that $t_n \notin N \cup J$ for all $n \geq 1$. By the previous step, $\partial_x \mathcal{E}(t_n, x(t_n)) \in \{-1, 1\}$ for all $n \geq 1$. Moreover, since $x(\cdot)$ is continuous at t , we get

$$\partial_x \mathcal{E}(t_n, x(t_n)) \rightarrow \partial_x \mathcal{E}(t, x(t)).$$

Therefore, $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$. □

As an easy consequence of Lemma 8, we have

Lemma 9. Assume that $x : [0, T] \rightarrow \mathbb{R}$ has bounded variation and satisfies the weak local stability (1) and the energy-dissipation upper bound (2). If $t \notin J \cup \text{int}(N \cup J)$ and $\partial_{xx} \mathcal{E}(t, x(t)) \neq 0$, then for any sequence $t_n \rightarrow t$ such that $t_n \notin J \cup \text{int}(N \cup J)$ and $t_n \neq t$ for all $n \geq 1$, one has

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathcal{E}(t, x(t))}{\partial_{xx} \mathcal{E}(t, x(t))}.$$

Here J is the jump set of $x(\cdot)$, and $N := \{t \in (0, T) \mid x'(t) = 0\}$.

Proof. By Lemma 8 we have $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$ and $\partial_x \mathcal{E}(t_n, x(t_n)) \in \{-1, 1\}$ for all $n \geq 1$. Due to the continuity of the function $s \mapsto \partial_x \mathcal{E}(s, x(s))$ at $s = t$, we obtain

$$\partial_x \mathcal{E}(t_n, x(t_n)) = \partial_x \mathcal{E}(t, x(t))$$

for n large enough. Therefore, by Taylor's expansion,

$$\begin{aligned} 0 &= \partial_x \mathcal{E}(t_n, x(t_n)) - \partial_x \mathcal{E}(t, x(t)) \\ &= \partial_{xt} \mathcal{E}(t, x(t)) \cdot (t_n - t) + \partial_{xx} \mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(t_n - t) + o(x(t_n) - x(t)), \end{aligned}$$

we get

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathcal{E}(t, x(t))}{\partial_{xx} \mathcal{E}(t, x(t))}.$$

Here we have assumed that $\partial_{xx}\mathcal{E}(t, x(t)) \neq 0$. □

Now we are able to give

Proof of Lemma 7. Let J be the jump set of $x(\cdot)$, $N := \{t \in (0, T) \mid x'(t) = 0\}$ and $E = \{t \in (0, T) \mid \partial_{xx}\mathcal{E}(t, x(t)) = 0\}$. By Assumption (H1) and by dividing the interval $[0, T]$ to be many smaller intervals if necessary, we have that $\partial_{xxx}\mathcal{E}(t, x(t)) \neq 0$ for any $t \in E$. For an arbitrary point $t \in (0, T)$, we have one of the following cases.

Case 1. If $t \in N \cup J$, then $t \notin B$, by the definition of B .

Case 2. If t is an accumulation point of $(0, T) \setminus (N \cup J)$ and $t \notin E$, then we can find a sequence $t_n \rightarrow t$ such that $t_n \notin N \cup J$ and $t_n \neq t$ for all $n \geq 1$. By Lemma 9,

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))}.$$

Thus in this case, $t \notin B$.

Case 3. If $t \notin J$ and t is an accumulation point of E , then we can find a sequence $s_n \in E$, $s_n \rightarrow t$. Using Taylor's expansion again, we get

$$\begin{aligned} 0 &= \partial_{xx}\mathcal{E}(s_n, x(s_n)) - \partial_{xx}\mathcal{E}(t, x(t)) \\ &= \partial_{xxt}\mathcal{E}(t, x(t)) \cdot (s_n - t) + \partial_{xxx}\mathcal{E}(t, x(t)) \cdot (x(s_n) - x(t)) \\ &\quad + o(s_n - t) + o(x(s_n) - x(t)). \end{aligned}$$

Since $\partial_{xxx}\mathcal{E}(t, x(t)) \neq 0$, we arrive at

$$\lim_{n \rightarrow \infty} \frac{x(s_n) - x(t)}{s_n - t} = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))},$$

which is a finite number. Thus $t \notin B$.

Conclusion. In summary, if $t \in B$, then either t is an isolated point of $(0, T) \setminus (N \cup J)$, or t is an isolated point of E . Therefore, B is at most countable. Since $\mu_s(\{t\}) = 0$ for any $t \in B \subset (0, T) \setminus J$, we have $|\mu_s|(B) = 0$. This ends the proof of Lemma 7. □

The proof of Theorem 2 is completed.

5. PROOF OF THEOREM 3

In this section we shall prove Theorem 3. We shall always assume that \mathcal{E} is of class C^3 . We shall also denote by J the jump set of $x(\cdot)$,

$$N := \{t \in (0, T) \mid x'(t) = 0\}$$

and

$$E := \{t \in (0, T) \mid \partial_{xx}\mathcal{E}(t, x(t)) = 0\}.$$

5.1. Proof of Theorem 3 (ii). To prove Theorem 3 (ii), besides Lemma 8 and Lemma 9, we need some other preliminary results.

Lemma 10. *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). If $t \notin J \cup \text{int}(N \cup J)$, then $\partial_{xx}\mathcal{E}(t, x(t)) \geq 0$. Moreover, if $t \in E$, then $\partial_x\mathcal{E}(t, x(t)) \cdot \partial_{xxx}\mathcal{E}(t, x(t)) \leq 0$.*

Proof. Step 1. By Lemma 8, we have $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$ for all $t \notin J \cup \text{int}(N \cup J)$. On the other hand, from the strong local stability (5), by using Taylor's expansion for $\mathcal{E}(t, \cdot)$ up to the second order, we can write

$$\begin{aligned} \mathcal{E}(t, x(t)) &\leq \mathcal{E}(t, x(t)) + |z - x(t)| + \partial_x \mathcal{E}(t, x(t)) \cdot (z - x(t)) \\ &\quad + \partial_{xx} \mathcal{E}(t, x(t)) \cdot \frac{(z - x(t))^2}{2} + o(|z - x(t)|^2) \end{aligned} \quad (8)$$

for z near $x(t)$. If $\partial_x \mathcal{E}(t, x(t)) = -1$, then taking a sequence $z_n \downarrow x(t)$ in (8) we get $\partial_{xx} \mathcal{E}(t, x(t)) \geq 0$. If $\partial_x \mathcal{E}(t, x(t)) = 1$, then taking a sequence $z_n \uparrow x(t)$ in (8), we also get $\partial_{xx} \mathcal{E}(t, x(t)) \geq 0$.

Step 2. Now assuming $\partial_{xx} \mathcal{E}(t, x(t)) = 0$, we shall prove that $\partial_x \mathcal{E}(t, x(t)) \cdot \partial_{xxx} \mathcal{E}(t, x(t)) \leq 0$. Using the above stability and Taylor's expansion for $\mathcal{E}(t, \cdot)$ up to the third order, we get

$$\begin{aligned} \mathcal{E}(t, x(t)) &\leq \mathcal{E}(t, x(t)) + |x - x(t)| + \partial_x \mathcal{E}(t, x(t)) \cdot (x - x(t)) \\ &\quad + \partial_{xxx} \mathcal{E}(t, x(t)) \cdot \frac{(x - x(t))^3}{6} + o(|x - x(t)|^3). \end{aligned} \quad (9)$$

If $\partial_x \mathcal{E}(t, x(t)) = -1$, then taking a sequence $x_n \downarrow x(t)$ in (9) we get $\partial_{xxx} \mathcal{E}(t, x(t)) \geq 0$. If $\partial_x \mathcal{E}(t, x(t)) = 1$, then taking a sequence $x_n \uparrow x(t)$ in (9), we get $\partial_{xxx} \mathcal{E}(t, x(t)) \leq 0$. Thus in both cases, we always have $\partial_x \mathcal{E}(t, x(t)) \cdot \partial_{xxx} \mathcal{E}(t, x(t)) \leq 0$. \square

Lemma 11. Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ satisfies the weak local stability (1) and the energy-dissipation upper bound (2). Then for all $t \in (0, T) \setminus J$, one has

$$\limsup_{s \rightarrow t} \left\{ \partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(s) - x(t)}{s - t} \right\} \leq 0.$$

Here J is the jump set of $x(\cdot)$.

Proof. We shall show that for any sequence $t_n \rightarrow t$ and $t_n \neq t$ then

$$\limsup_{n \rightarrow \infty} \left\{ \partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{t_n - t} \right\} \leq 0.$$

Of course, we may assume that

$$\liminf_{n \rightarrow \infty} \left| \frac{x(t_n) - x(t)}{t_n - t} \right| > 0$$

and either $t_n \downarrow t$ or $t_n \uparrow t$.

Case 1. If $t_n \downarrow t$, then due to the inequality (7) in the proof of Lemma 8 one has

$$\lim_{n \rightarrow \infty} \partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} = -1.$$

This implies that

$$\lim_{n \rightarrow \infty} \partial_x \mathcal{E}(t, x(t)) \cdot \text{sign} \left(\frac{x(t_n) - x(t)}{t_n - t} \right) = -1.$$

Case 2. If $t_n \uparrow t$, then similarly, one has

$$\lim_{n \rightarrow \infty} \partial_x \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{|x(t_n) - x(t)|} = 1,$$

and hence

$$\lim_{n \rightarrow \infty} \partial_x \mathcal{E}(t, x(t)) \cdot \text{sign} \left(\frac{x(t_n) - x(t)}{t_n - t} \right) = -1.$$

Thus in all cases, we have

$$\lim_{n \rightarrow \infty} \partial_x \mathcal{E}(t, x(t)) \cdot \text{sign} \left(\frac{x(t_n) - x(t)}{t_n - t} \right) = -1.$$

and the conclusion follows. \square

Lemma 12. *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). Let $t \in \text{int}[(0, T) \setminus \text{int}(N)]$ such that $\partial_{xx} \mathcal{E}(t, x(t)) = 0$ and $\partial_{xxx} \mathcal{E}(t, x(t)) \neq 0$. Then $\partial_{xt} \mathcal{E}(t, x(t)) = 0$.*

Proof. Step 1. Take an arbitrary sequence $t_n \rightarrow t$, $t_n \neq t$, $t_n \in \text{int}[(0, T) \setminus \text{int}(N)]$. By Lemma 8 and the continuity of the function $s \mapsto \partial_x \mathcal{E}(s, x(s))$, we have

$$\partial_x \mathcal{E}(t_n, x(t_n)) = \partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$$

for all n large enough. Using Taylor's expansion and the assumption $\partial_{xx} \mathcal{E}(t, x(t)) = 0$, we have

$$\begin{aligned} 0 &= \partial_x \mathcal{E}(t_n, x(t_n)) - \partial_x \mathcal{E}(t, x(t)) \\ &= \partial_{xt} \mathcal{E}(t, x(t)) \cdot (t_n - t) + o(x(t_n) - x(t)) + o(t_n - t). \end{aligned} \quad (10)$$

Thus we can conclude that $\partial_{xt} \mathcal{E}(t, x(t)) = 0$ if we can find a sequence $t_n \rightarrow t$ such that

$$\limsup_{n \rightarrow \infty} \frac{|x(t_n) - x(t)|}{|t_n - t|} < \infty.$$

Step 2. For an arbitrary sequence $s_n \rightarrow t$, $s_n \neq t$, $s_n \in \text{int}[(0, T) \setminus \text{int}(N)]$, by Lemma 10 we have

$$\partial_{xx} \mathcal{E}(s_n, x(s_n)) \geq 0 = \partial_{xx} \mathcal{E}(t, x(t)).$$

Therefore, using Taylor's expansion we obtain

$$\begin{aligned} 0 &\leq \partial_{xx} \mathcal{E}(s_n, x(s_n)) - \partial_{xx} \mathcal{E}(t, x(t)) \\ &= \partial_{xxt} \mathcal{E}(t, x(t)) \cdot (s_n - t) + \partial_{xxx} \mathcal{E}(t, x(t)) \cdot (x(s_n) - x(t)) \\ &\quad + o(x(s_n) - x(t)) + o(s_n - t). \end{aligned}$$

Choosing $s_n \uparrow t$ and dividing the above inequality for $(s_n - t) < 0$, we have

$$0 \geq \partial_{xxt} \mathcal{E}(t, x(t)) + (\partial_{xxx} \mathcal{E}(t, x(t)) + o(1)) \cdot \frac{x(s_n) - x(t)}{s_n - t} + o(1). \quad (11)$$

Step 3. Since $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$, we distinguish two cases.

Case 1. Assume $\partial_x \mathcal{E}(t, x(t)) = -1$. Then $\partial_{xxx} \mathcal{E}(t, x(t)) > 0$ by Lemma 10. Therefore, (11) implies that

$$\limsup_{n \rightarrow \infty} \frac{x(s_n) - x(t)}{s_n - t} \leq -\frac{\partial_{xxt} \mathcal{E}(t, x(t))}{\partial_{xxx} \mathcal{E}(t, x(t))} < \infty.$$

On the other hand, by Lemma 11,

$$\liminf_{n \rightarrow \infty} \left\{ \frac{x(s_n) - x(t)}{s_n - t} \right\} \geq 0.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Case 2. Assume $\partial_x \mathcal{E}(t, x(t)) = 1$. Similarly, we have $\partial_{xxx} \mathcal{E}(t, x(t)) < 0$ by Lemma 10, and hence

$$\liminf_{n \rightarrow \infty} \frac{x(s_n) - x(t)}{s_n - t} \geq -\frac{\partial_{xxt} \mathcal{E}(t, x(t))}{\partial_{xxx} \mathcal{E}(t, x(t))} > -\infty.$$

Moreover, by Lemma 11,

$$\limsup_{n \rightarrow \infty} \left\{ \frac{x(s_n) - x(t)}{s_n - t} \right\} \leq 0.$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Step 4. In summary, if $s_n \uparrow t$, then we always have

$$\limsup_{n \rightarrow \infty} \frac{|x(s_n) - x(t)|}{|s_n - t|} < \infty.$$

Therefore, choosing $t_n = s_n$ in (10), we conclude that $\partial_{xt} \mathcal{E}(t, x(t)) = 0$. \square

Lemma 13. Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). If $t \in (0, T)$ is an accumulation point of $\partial \overset{\circ}{N}$, then $\partial_{xt} \mathcal{E}(t, x(t)) = 0$. Moreover, if $\partial_{xx} \mathcal{E}(t, x(t)) \neq 0$, then $x'(t) = 0$ and $\partial_{xtt} \mathcal{E}(t, x(t)) = 0$.

Here for convenience, we denote by $\partial \overset{\circ}{N}$ the boundary of $\text{int}(N)$.

Proof. **Step 1.** Since t is an accumulation point of $\partial \overset{\circ}{N}$, we can find $a_n \rightarrow t$, $b_n \rightarrow t$ such that $(a_n, b_n) \subset \text{int}(N)$ and $a_n, b_n \in \partial \overset{\circ}{N}$. By Lemma 8, and the continuity of $s \mapsto \partial_x \mathcal{E}(s, x(s))$, one has, for n large enough,

$$\partial_x \mathcal{E}(a_n, x(a_n)) = \partial_x \mathcal{E}(t, x(t)) = \partial_x \mathcal{E}(b_n, x(b_n)) \in \{-1, 1\}.$$

Note that for all $s \in [a_n, b_n]$, $x(s) = c_n$, a constant independent of s . Consider the one-variable function

$$s \mapsto f_n(s) := \partial_x \mathcal{E}(s, c_n).$$

Since $f_n(a_n) = f_n(b_n)$, by Rolle's Theorem, we can find a number $s_n \in (a_n, b_n)$ such that $f'_n(s_n) = 0$. This means $\partial_{xt} \mathcal{E}(s_n, x(s_n)) = 0$. Since $s_n \rightarrow t$, one has

$$0 = \partial_{xt} \mathcal{E}(s_n, x(s_n)) \rightarrow \partial_{xt} \mathcal{E}(t, x(t)).$$

Step 2. Now we assume that $t \notin E$. We distinguish two cases.

Case 1. Let $t_n \notin \text{int}(N)$, $t_n \neq t$ and $t_n \rightarrow t$. Then by Lemma 9 we have

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xt} \mathcal{E}(t, x(t))}{\partial_{xx} \mathcal{E}(t, x(t))} = 0.$$

Case 2. Let $s_n \in \text{int}(N)$ and $s_n \rightarrow t$. Since t is an accumulation point of $\partial \overset{\circ}{N}$, we can assume that $s_n \in (a_n, b_n) \subset \text{int}(N)$ with $a_n, b_n \in \partial \overset{\circ}{N}$. Using Case 1, one has

$$\lim_{n \rightarrow \infty} \frac{x(a_n) - x(t)}{a_n - t} = \lim_{n \rightarrow \infty} \frac{x(b_n) - x(t)}{b_n - t} = 0.$$

On the other hand, since $x'(s) = 0$ when $s \in (a_n, b_n)$, we have $x(s_n) = x(a_n) = x(b_n)$. Therefore,

$$\left| \frac{x(s_n) - x(t)}{s_n - t} \right| \leq \max \left\{ \left| \frac{x(a_n) - x(t)}{a_n - t} \right|, \left| \frac{x(b_n) - x(t)}{b_n - t} \right| \right\} \rightarrow 0$$

as $n \rightarrow \infty$.

Thus in summary, for any sequence $t_n \rightarrow t$ and $t_n \neq t$ we always have

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} \rightarrow -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))} = 0.$$

This means $x'(t) = 0$.

Step 3. Now we show that if we assume furthermore that $t \notin E$, then $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$.

Since $t \notin E$ and the function $s \mapsto \partial_{xx}\mathcal{E}(s, x(s))$ is continuous at $s = t$, we have $s \notin E$ if s is in a neighborhood of t (recall that E is closed). In particular, if s is in a neighborhood of t and $s \in \text{int}[(0, T) \setminus \text{int}(N)]$, then $\partial_{xx}\mathcal{E}(s, x(s)) > 0$ by Lemma 10. Moreover, if $s \notin J$, then

$$x'(s) = -\frac{\partial_{xt}\mathcal{E}(s, x(s))}{\partial_{xx}\mathcal{E}(s, x(s))}$$

by Lemma 9. Using Lemma 11, we conclude that

$$\partial_{xt}\mathcal{E}(s, x(s)) \cdot \partial_x\mathcal{E}(s, x(s)) \geq 0. \quad (12)$$

Let us assume that $\partial_x\mathcal{E}(t, x(t)) = -1$ (the other case, $\partial_x\mathcal{E}(t, x(t)) = 1$, can be treated by the same way). If s is in a neighborhood of t , $s \notin J$ and $s \in \text{int}[(0, T) \setminus \text{int}(N)]$, then $\partial_x\mathcal{E}(s, x(s)) < 0$, and hence $\partial_{xt}\mathcal{E}(s, x(s)) \leq 0$ by (12). Using the continuity of $s \mapsto \partial_{xt}\mathcal{E}(s, x(s))$, we have

$$\partial_{xt}\mathcal{E}(a_n, x(a_n)) \leq 0 \text{ and } \partial_{xt}\mathcal{E}(b_n, x(b_n)) \leq 0$$

for n large enough, where $\{a_n\}, \{b_n\}$ are taken as in Step 1.

On the other hand, it was already shown in Step 1 that there exists $t_n \in (a_n, b_n)$ such that $\partial_{xt}\mathcal{E}(t_n, x(t_n)) = 0$. Therefore, the function $g(s) := \partial_{xt}\mathcal{E}(s, x(s))$ has a local maximizer $s_n \in (a_n, b_n)$. Therefore, for n large enough,

$$\partial_{xtt}\mathcal{E}(s_n, x(s_n)) = g'(s_n) = 0.$$

Since $s_n \rightarrow t$, by taking the limit as $n \rightarrow \infty$ we obtain $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$. \square

Now we are able to give

Proof of Theorem 3 (ii). Since $x(\cdot)$ has finite jump points and (H1), (H3) hold true, by dividing $(0, T)$ into the subintervals if necessary, we may further assume that $x(\cdot)$ is continuous on $[0, T]$ and

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x\mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx}\mathcal{E}(t, x) = \partial_{xxx}\mathcal{E}(t, x) = 0\} = \emptyset,$$

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x\mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx}\mathcal{E}(t, x) = \partial_{xt}\mathcal{E}(t, x) = 0\} = \emptyset.$$

We denote by I_1 the set of isolated points of ∂N . It remains to consider when $t \notin I_1$. We distinguish the following cases.

Case 1. If $t \in \text{int}(N)$, then $x'(t) = 0$. Moreover, if s is in a neighborhood of t then $x'(s) = 0$. Therefore, $x'(\cdot)$ is continuous at t .

Case 2. If $t \in \text{int}[(0, T) \setminus \text{int}(N)]$, then by Lemma 12 we have $t \notin E$. Therefore, by Lemma 9,

$$x'(t) = -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))}.$$

Since the same formula also holds true for any s in a neighborhood of t , we have that $x'(\cdot)$ is continuous at t .

Case 3. If t is an accumulation point of $\partial \overset{\circ}{N}$, then $\partial_{xt}\mathcal{E}(t, x(t)) = 0$ by Lemma 13. Therefore, $t \notin E$. By Lemma 13 one has $x'(t) = 0$. Next, we shall show that if $t_n \rightarrow t$ and $t_n \notin I_1$, then $x'(t_n) \rightarrow x'(t) = 0$. Indeed, if $t_n \in \text{int}[(0, T) \setminus \text{int}(N)]$, then

$$x'(t_n) = -\frac{\partial_{xt}\mathcal{E}(t_n, x(t_n))}{\partial_{xx}\mathcal{E}(t_n, x(t_n))} \rightarrow -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))} = 0.$$

Otherwise, if $t_n \in \text{int}(N)$ or t_n is an accumulation point of $\partial \overset{\circ}{N}$, then we already have $x'(t_n) = 0$.

In summary, if $t \in (0, T) \setminus I_1$ one has

$$x'(t) = \begin{cases} -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))} & \text{if } t \in \text{int}[(0, T) \setminus \text{int}(N)], \\ 0 & \text{otherwise,} \end{cases}$$

and $x'(\cdot)$ is continuous on $(0, T) \setminus I_1$. This completes the proof of Theorem 3 (ii). \square

Remark. In general (when the jump set of $x(\cdot)$ and the sets in (H1) and (H3) are finite, instead of empty), the set I in the statement of Theorem 3 (ii) contains the following points: the isolated points of $\partial \overset{\circ}{N}$ (namely the set I_1 in the above proof), the jump points, and the points t such that $\partial_x\mathcal{E}(t, x(t)) \in \{-1, 1\}$, $\partial_{xx}\mathcal{E}(t, x(t)) = 0$ and either $\partial_{xxx}\mathcal{E}(t, x(t)) = 0$ or $\partial_{xt}\mathcal{E}(t, x(t)) = 0$.

5.2. Proof of Theorem 3 (iii). To prove Theorem 3 (iii), we need the further preliminary lemmas.

Lemma 14. Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). Assume furthermore that

$$t \notin \text{int}(N), \partial_{xx}\mathcal{E}(t, x(t)) = \partial_{xt}\mathcal{E}(t, x(t)) = 0 \text{ and } \partial_{xxx}\mathcal{E}(t, x(t)) \neq 0.$$

Then the limits

$$\lim_{s \notin \text{int}(N), s \downarrow t} \frac{x(s) - x(t)}{s - t} \text{ and } \lim_{s \notin \text{int}(N), s \uparrow t} \frac{x(s) - x(t)}{s - t},$$

exist and they are two solutions to the equation (w.r.t. X)

$$\partial_{xtt}\mathcal{E}(t, x(t)) + 2\partial_{xxt}\mathcal{E}(t, x(t)) \cdot X + \partial_{xxx}\mathcal{E}(t, x(t)) \cdot X^2 = 0. \quad (13)$$

Moreover, if there is a sequence $t_n \rightarrow t$ such that

$$t_n \neq t, t_n \notin \text{int}(N) \text{ and } \partial_{xx}\mathcal{E}(t_n, x(t_n)) = 0$$

for all $n \geq 1$, then the equation (13) has a unique solution $X = -\partial_{xxt}\mathcal{E}(t, x(t))/\partial_{xxx}\mathcal{E}(t, x(t))$.

Here recall that $N := \{t \in (0, T) \mid x'(t) = 0\}$.

Proof. Step 1. Let $t_n \rightarrow t$ and $t_n \notin \text{int}(N)$. We have $\partial_x \mathcal{E}(t_n, x(t_n)) = \partial_x \mathcal{E}(t, x(t))$ by Lemma 8 and the continuity of $s \mapsto \partial_x \mathcal{E}(s, x(s))$ at $s = t$. Using Taylor's expansion we obtain

$$\begin{aligned} 0 &= \partial_x \mathcal{E}(t_n, x(t_n)) - \partial_x \mathcal{E}(t, x(t)) \\ &= \partial_{xtt} \mathcal{E}(t, x(t)) \cdot (t_n - t)^2 + 2\partial_{xxt} \mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) \cdot (t_n - t) \\ &\quad + \partial_{xxx} \mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t))^2 + o(|x(t_n) - x(t)|^2) + o(|t_n - t|^2). \end{aligned}$$

Dividing this equality by $(t_n - t)^2$ and taking the limit as $n \rightarrow \infty$ we get

$$\partial_{xtt} \mathcal{E}(t, x(t)) + 2\partial_{xxt} \mathcal{E}(t, x(t)) \cdot \frac{x(t_n) - x(t)}{t_n - t} + \partial_{xxx} \mathcal{E}(t, x(t)) \cdot \left(\frac{x(t_n) - x(t)}{t_n - t} \right)^2 \rightarrow \mathbf{(14)}$$

Notice that, (14) also shows that the solutions of (13) are real. Moreover, if we denote by X_1 and X_2 the two solutions of the equation (13), then

$$\min \left\{ \left| \frac{x(t_n) - x(t)}{t_n - t} - X_1 \right|, \left| \frac{x(t_n) - x(t)}{t_n - t} - X_2 \right| \right\} \rightarrow 0$$

as $n \rightarrow \infty$.

Step 2. Using Lemma 10 and Taylor's expansion one has

$$\begin{aligned} 0 &\leq \partial_{xx} \mathcal{E}(t_n, x(t_n)) - \partial_{xx} \mathcal{E}(t, x(t)) \\ &= \partial_{xxt} \mathcal{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxx} \mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) \\ &\quad + o(t_n - t) + o(x(t_n) - x(t)). \end{aligned} \tag{15}$$

By Lemma 8, $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$. We distinguish two cases.

Case 1. $\partial_x \mathcal{E}(t, x(t)) = -1$. In this case, by Lemma 10 we have $\partial_{xxx} \mathcal{E}(t, x(t)) > 0$. Therefore, from the inequality (15), if $t_n \downarrow t$, then

$$\liminf_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} \geq -\frac{\partial_{xxt} \mathcal{E}(t, x(t))}{\partial_{xxx} \mathcal{E}(t, x(t))},$$

while if $t_n \uparrow t$, then

$$\limsup_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} \leq -\frac{\partial_{xxt} \mathcal{E}(t, x(t))}{\partial_{xxx} \mathcal{E}(t, x(t))}.$$

Since

$$\max\{X_1, X_2\} \geq -\frac{\partial_{xxt} \mathcal{E}(t, x(t))}{\partial_{xxx} \mathcal{E}(t, x(t))} \geq \min\{X_1, X_2\},$$

the convergence in (14) reduces to

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \max\{X_1, X_2\} \text{ if } t_n \downarrow t,$$

and

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \min\{X_1, X_2\} \text{ if } t_n \uparrow t.$$

Case 2. If $\partial_x \mathcal{E}(t, x(t)) = 1$, then similarly,

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \min\{X_1, X_2\} \text{ if } t_n \downarrow t,$$

and

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \max\{X_1, X_2\} \text{ if } t_n \uparrow t.$$

In both cases, the first conclusion of Lemma 14 follows.

Step 3. Now assume that there is a sequence $t_n \rightarrow t$ such that $t_n \neq t$, $t_n \notin \text{int}(N)$ and $\partial_{xx}\mathcal{E}(t_n, x(t_n)) = 0$ for all $n \geq 1$. Using Taylor's expansion ,

$$\begin{aligned} 0 &= \partial_{xx}\mathcal{E}(t_n, x(t_n)) - \partial_{xx}\mathcal{E}(t, x(t)) \\ &= \partial_{xxt}\mathcal{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxx}\mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) \\ &\quad + o(t_n - t) + o(x(t_n) - x(t)), \end{aligned}$$

we find that

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))}.$$

Thus $-\partial_{xxt}\mathcal{E}(t, x(t))/\partial_{xxx}\mathcal{E}(t, x(t))$ is a solution to (13). Substituting this solution into (13) we find that

$$[\partial_{xxt}\mathcal{E}(t, x(t))]^2 = \partial_{xtt}\mathcal{E}(t, x(t)) \cdot \partial_{xxx}\mathcal{E}(t, x(t)),$$

which implies that (13) has a unique solution. \square

Lemma 15. Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). Let t be an accumulation point of $\partial \overset{\circ}{N}$, and assume either $t \notin E$, or $t \in E$ and $\partial_{xxx}\mathcal{E}(t, x(t)) \neq 0$. Then $x'(t) = 0$, and $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$.

Here recall that $N := \{t \in (0, T) \mid x'(t) = 0\}$ and $E := \{t \in (0, T) \mid \partial_{xx}\mathcal{E}(t, x(t)) = 0\}$.

Proof. Since t is an accumulation point of $\partial \overset{\circ}{N}$, Lemma 13 ensures that $\partial_{xt}\mathcal{E}(t, x(t)) = 0$. If $t \notin E$, then Lemma 13 also implies that $x'(t) = 0$ and $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$. Therefore, it remains to consider the case $t \in E$.

Step 1. Since t is an accumulation point of $\partial \overset{\circ}{N}$, there exists a sequence $\{(a_n, b_n)\}$ such that $(a_n, b_n) \subset \text{int}(N)$, $a_n, b_n \in \partial \overset{\circ}{N}$ for all $n \geq 1$, and $a_n, b_n \downarrow t$ (or $a_n, b_n \uparrow t$). By Lemma 14, we have

$$\lim_{n \rightarrow \infty} \frac{x(a_n) - x(t)}{a_n - t} = \lim_{n \rightarrow \infty} \frac{x(b_n) - x(t)}{b_n - t} = X_1,$$

where X_1 is a solution to (13). Note that $x(s) = c_n$, a constant, when $s \in [a_n, b_n]$. Therefore, if $t_n \in [a_n, b_n]$ for all $n \geq 1$, then using the fact that $x(\cdot)$ is a constant in $[a_n, b_n]$, one has

$$\left| \frac{x(t_n) - x(t)}{t_n - t} - X_1 \right| \leq \max \left\{ \left| \frac{x(b_n) - x(t)}{b_n - t} - X_1 \right|, \left| \frac{x(a_n) - x(t)}{a_n - t} - X_1 \right| \right\} \rightarrow 0.$$

Thus if $t_n \in [a_n, b_n]$, then

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = X_1.$$

Step 2. On the other hand, by Lemma 8 and the continuity of $s \mapsto \partial_x \mathcal{E}(s, x(s))$ at $s = t$, we have

$$\partial_x \mathcal{E}(a_n, x(a_n)) = \partial_x \mathcal{E}(t, x(t)) = \partial_x \mathcal{E}(b_n, x(b_n)) \in \{-1, 1\} \quad (16)$$

for n large enough. Consider the one-variable function

$$f_n(s) := \partial_x \mathcal{E}(s, c_n) \text{ on } s \in [a_n, b_n], \quad (17)$$

where recall that $x(s) = c_n$ for all $s \in [a_n, b_n]$. Since $f_n(a_n) = f_n(b_n)$, by applying Rolle's Theorem, we can find $t_n \in (a_n, b_n)$ such that

$$\partial_{xt}\mathcal{E}(t_n, x(t_n)) = f'_n(t_n) = 0.$$

Using Taylor's expansion we have

$$\begin{aligned} 0 &= \partial_{xt}\mathcal{E}(t_n, x(t_n)) - \partial_{xt}\mathcal{E}(t, x(t)) \\ &= \partial_{xtt}\mathcal{E}(t, x(t)) \cdot (t_n - t) + \partial_{xxt}\mathcal{E}(t, x(t)) \cdot (x(t_n) - x(t)) + o(t_n - t) + o(x(t_n) - x(t)). \end{aligned}$$

Dividing this equality by $t_n - t$ and taking the limit as $n \rightarrow \infty$ we obtain

$$\partial_{xtt}\mathcal{E}(t, x(t)) + \partial_{xxt}\mathcal{E}(t, x(t)) \cdot X_1 = 0. \quad (18)$$

Step 3. We show that $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$. Assume by contradiction that $\partial_{xtt}\mathcal{E}(t, x(t)) \neq 0$. Then from (18), we must have $\partial_{xxt}\mathcal{E}(t, x(t)) \neq 0$ and

$$X_1 = -\frac{\partial_{xtt}\mathcal{E}(t, x(t))}{\partial_{xxt}\mathcal{E}(t, x(t))} \neq 0.$$

Since X_1 is a solution to (13), we obtain

$$[\partial_{xxt}\mathcal{E}(t, x(t))]^2 = \partial_{xtt}\mathcal{E}(t, x(t)) \cdot \partial_{xxx}\mathcal{E}(t, x(t)), \quad (19)$$

which in particular implies that X_1 is the unique solution to (13).

From (16), let us assume that

$$\partial_x\mathcal{E}(a_n, x(a_n)) = \partial_x\mathcal{E}(t, x(t)) = \partial_x\mathcal{E}(b_n, x(b_n)) = -1$$

for n large enough (the other case can be treated by the same way).

By Lemma 10 one has $\partial_{xxx}\mathcal{E}(t, x(t)) > 0$. From (19) one has $\partial_{xtt}\mathcal{E}(t, x(t)) > 0$. By the continuity of $s \mapsto \partial_{xtt}\mathcal{E}(s, x(s))$ at $s = t$, we have $\partial_{xtt}\mathcal{E}(s, x(s)) > 0$ when s is in a neighborhood of t . In particular, the function $f_n(s)$ defined by (17) satisfies

$$f''_n(s) = \partial_{xtt}\mathcal{E}(s, x(s)) > 0 \text{ for all } s \in (a_n, b_n)$$

for n large enough.

Thus f_n is strictly convex on $[a_n, b_n]$. Consequently, if we choose $s := (a_n + b_n)/2$, then

$$\partial_x\mathcal{E}(s, x(s)) = f_n(s) < \frac{f(a_n) + f(b_n)}{2} = -1.$$

However, this contradicts to the fact that $\partial_x\mathcal{E}(s, x(s)) \geq -1$ for all $s \notin \text{int}(N)$ by Lemma 8. Thus we must have $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$.

Step 4. Now we show that $X_1 = 0$. In fact, if $\partial_{xxt}\mathcal{E}(t, x(t)) \neq 0$, then from $\partial_{xtt}\mathcal{E}(t, x(t)) = 0$ and (18) we must have $X_1 = 0$. Otherwise, if $\partial_{xxt}\mathcal{E}(t, x(t)) = 0$, then 0 is the unique solution to the equation (13), and hence we also have $X_1 = 0$.

Step 5. Now we show that $x'(t) = 0$. We distinguish three cases.

Case 1. Assume that there exists $a < t$ such that $(a, t) \subset \text{int}(N)$. It is obvious that $x'_-(t) = 0 = \lim_{s \uparrow t} x'(s)$. It remains to show that $x'_+(t) = 0$, namely to show that

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = 0$$

provided that $t_n \downarrow t$.

First, we assume that $t_n \in \text{int}(N)$ and $t_n \downarrow t$. Note that $(t, b) \not\subset \text{int}(N)$ for all $b > t$ (otherwise, by the continuity we have $x(a) = x(t) = x(b)$ and $t \in (a, b) \subset \text{int}(N)$, which is a contradiction). Therefore, as in Step 1, we can choose the sequence $\{(a_n, b_n)\}$ such that $(a_n, b_n) \subset \text{int}(N)$, $a_n, b_n \in \partial \overset{\circ}{N}$ for all $n \geq 1$, and $a_n, b_n \downarrow t$. Therefore, it follows from Step 1 and the fact that $X_1 = 0$

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n \rightarrow \infty} \frac{x(a_n) - x(t)}{a_n - t} = 0.$$

Next, assume that $t_n \notin \text{int}(N)$ and $t_n \downarrow t$. Then by Lemma 14 we have

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n \rightarrow \infty} \frac{x(a_n) - x(t)}{a_n - t} = 0.$$

Thus for any sequence $t_n \downarrow t$ we obtain

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = 0.$$

Therefore, $x'_+(t) = 0$. Thus $x'(t) = 0$.

Case 2. If there exists $b > t$ such that $(t, b) \subset \text{int}(N)$, then similarly to Case 1 we have $x'(t) = 0$.

Case 3. Finally, assume that $(a, t) \not\subset \text{int}(N)$ for all $a < t$, and $(t, b) \not\subset \text{int}(N)$ for all $b > t$. Then by the same proof in Case 1, using the fact that $(t, b) \not\subset \text{int}(N)$ for all $b > t$, we have $x'_+(t) = 0$. Similarly, using the fact that $(a, t) \not\subset \text{int}(N)$ for all $a < t$, we obtain $x'_-(t) = 0$. Thus $x'(t) = 0$. This completes our proof. \square

Now we are able to give

Proof of Theorem 3 (iii). **Step 1.** Since $x(\cdot)$ has only finite jumps and (H1), (H3), (H4) hold true, by dividing $(0, T)$ into subintervals if necessary, we may assume that $x(\cdot)$ has no jump and

$$\begin{aligned} \{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xxx} \mathcal{E}(t, x) = 0\} &= \emptyset, \\ \{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xt} \mathcal{E}(t, x) = 0\} &= \emptyset, \\ \{(t, x) \in [0, T] \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xt} \mathcal{E}(t, x) = \partial_{xtt} \mathcal{E}(t, x) = 0\} &= \emptyset. \end{aligned}$$

Step 2. Assume that $\partial \overset{\circ}{N}$ has an accumulation point t . Then we have $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$ by Lemma 8. Note that if $t = 0$ or $t = T$, then Lemma 8 is not applicable directly to t , but because t is an accumulation point of $\partial \overset{\circ}{N}$, we can apply Lemma 8 to the points in $\partial \overset{\circ}{N} \cap (0, T)$ first, and then take the limit to get the conclusion at t .

Next, we have $\partial_{xt} \mathcal{E}(t, x(t)) = 0$ by Lemma 13, and $\partial_{xtt} \mathcal{E}(t, x(t)) = 0$ by Lemma 13 (when $t \notin E$) and Lemma 15 (when $t \in E$). Note that these lemmas apply even if $t = 0$ or $t = T$.

Thus

$$\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}, \partial_{xt} \mathcal{E}(t, x(t)) = 0, \partial_{xtt} \mathcal{E}(t, x(t)) = 0.$$

By condition (H3), this case cannot happen. Therefore, $\partial \overset{\circ}{N}$ has no accumulation point. Thus $\partial \overset{\circ}{N}$ is finite, and hence $\text{int}(N) \cup \text{int}[(0, T) \setminus \text{int}(N)]$ is the union of finitely many open intervals.

Step 3. Finally, if $t \in \text{int}(N)$, then $x'(t) = 0$. On the other hand, if $t \in \text{int}[(0, T) \setminus \text{int}(N)]$, then by Lemma 12 we have $t \notin E$, and hence

$$x'(t) = -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))}$$

by Lemma 9. Thus we can conclude that $x(\cdot)$ is of class C^1 in $\text{int}(N) \cup \text{int}[(0, T) \setminus \text{int}(N)]$. The proof is completed. \square

5.3. Proof of Theorem 3 (i). Finally, to obtain Theorem 3 (i), we need the following lemma.

Lemma 16. *Assume that the BV function $x : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies the strong local stability (5) and the energy-dissipation upper bound (2). If $t \in \text{int}[(0, T) \setminus \text{int}(N)]$, $t \in E$ and $\partial_{xxx}\mathcal{E}(t, x(t)) \neq 0$, then the right and left derivatives*

$$x'_+(t) := \lim_{s \downarrow t} \frac{x(s) - x(t)}{s - t}, \quad x'_-(t) := \lim_{s \uparrow t} \frac{x(s) - x(t)}{s - t},$$

exist and they are two solutions of the equation (13).

Moreover, if t is an accumulation point of E , then

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))}$$

and it is the unique solution to the equation (13).

On the other hand, if t is an isolated point of E , then either

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))},$$

or

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

Here recall that $N := \{t \in (0, T) \mid x'(t) = 0\}$ and $E := \{t \in (0, T) \mid \partial_{xx}\mathcal{E}(t, x(t)) = 0\}$.

Proof. Step 1. Since $t \in \text{int}[(0, T) \setminus \text{int}(N)]$ and $t \in E$, Lemma 12 ensures that $\partial_{xt}\mathcal{E}(t, x(t)) = 0$. Therefore, by Lemma 14, we get that $x'_+(t)$, $x'_-(t)$ exist and they are two solutions of the equation (13).

Step 2. If t is an accumulation point of E , then by Lemma 14 again, the equation (13) has a unique solution $-\partial_{xxt}\mathcal{E}(t, x(t))/\partial_{xxx}\mathcal{E}(t, x(t))$. Therefore,

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))}.$$

Step 3. Now we assume that t is an isolated point of E . If the equation (13) has a unique solution, then it must be $-\partial_{xxt}\mathcal{E}(t, x(t))/\partial_{xxx}\mathcal{E}(t, x(t))$, and hence

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))}.$$

Otherwise, if the equation (13) has two distinct solutions, then we shall show that

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

In fact, since t is an isolated point of E , when s is in a neighborhood of t we have $s \notin E$. Therefore, using Lemma 9 and L'Hopital's rule, we have, as $s \downarrow t$,

$$\begin{aligned} x'(s) &= -\frac{\partial_{xt}\mathcal{E}(s, x(s))}{\partial_{xx}\mathcal{E}(s, x(s))} = -\frac{\left(\frac{\partial_{xt}\mathcal{E}(s, x(s)) - \partial_{xt}\mathcal{E}(t, x(t))}{s-t}\right)}{\left(\frac{\partial_{xx}\mathcal{E}(s, x(s)) - \partial_{xx}\mathcal{E}(t, x(t))}{s-t}\right)} \\ &\rightarrow -\frac{\partial_{xtt}\mathcal{E}(t, x(t)) + \partial_{xxt}\mathcal{E}(t, x(t))x'_+(t)}{\partial_{xxt}\mathcal{E}(t, x(t)) + \partial_{xxx}\mathcal{E}(t, x(t))x'_+(t)} = x'_+(t). \end{aligned}$$

Here in the last identity we have used that $x'_+(t)$ solves the equation (13). Note that $\partial_{xxt}\mathcal{E}(t, x(t)) + \partial_{xxx}\mathcal{E}(t, x(t))x'_+(t) \neq 0$ because the equation (13) has two distinct solutions.

Similarly, as $s \uparrow t$,

$$x'(s) = -\frac{\partial_{xt}E(s, x(s))}{\partial_{xx}E(s, x(s))} \rightarrow x'_-(t).$$

The proof is completed. \square

Thus we can now provide

Proof of Theorem 3 (i). Step 1. Assume that $x(\cdot)$ has only finitely many jumps and (H1) holds. By dividing $(0, T)$ into subintervals if necessary, we may further assume that $x(\cdot)$ has no jumps and

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xxx} \mathcal{E}(t, x) = 0\} = \emptyset.$$

Thus either $t \notin E$, or $t \in E$ and $\partial_{xxx}\mathcal{E}(t, x(t)) \neq 0$. Choose I_3 and I_1 as follows

$$I_3 := \{t \in [0, T] \mid x(\cdot) \text{ is differentiable at } t\};$$

$$I_1 := \{t \in [0, T] \setminus I_3 \mid t \text{ is an isolated point of } \partial \overset{\circ}{N}\}.$$

Now we consider the case t is not an isolated point of $\partial \overset{\circ}{N}$. We have the following cases.

Case 1. If $t \in \text{int}(N)$, then $x'(t) = 0$ by definition.

Case 2. If t is an accumulation point of $\partial \overset{\circ}{N}$, then $x'(t) = 0$ by Lemma 15.

Case 3. If $t \in \text{int}[(0, T) \setminus \text{int}(N)]$ and $t \notin E$, then by Lemma 9,

$$x'(t) = -\frac{\partial_{xt}\mathcal{E}(t, x(t))}{\partial_{xx}\mathcal{E}(t, x(t))}.$$

Case 4. If $t \in \text{int}[(0, T) \setminus \text{int}(N)]$ and t is an accumulation point of E , then by Lemma 16,

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))}.$$

Case 5. If $t \in \text{int}[(0, T) \setminus \text{int}(N)]$ and t is an isolated point of E , then by Lemma 16, we have either

$$x'(t) = -\frac{\partial_{xxt}\mathcal{E}(t, x(t))}{\partial_{xxx}\mathcal{E}(t, x(t))},$$

or there exist $x'_+(t)$, $x'_-(t)$ and

$$x'_+(t) = \lim_{s \downarrow t} x'(s), \quad x'_-(t) = \lim_{s \uparrow t} x'(s).$$

Thus we can choose I_2 as follows

$$I_2 := \{t \in [0, T] \setminus (I_1 \cup I_3) \mid t \text{ is an isolated point of } E \text{ in } \text{int}[(0, T) \setminus \text{int}(N)] \text{ and } x'_-(t) \neq x'_+(t)\}.$$

Step 2. Assume that (H2) also holds. Then by dividing $(0, T)$ into subintervals again we may assume further that

$$\{(t, x) \in (0, T) \times \mathbb{R} \mid \partial_x \mathcal{E}(t, x) \in \{-1, 1\}, \partial_{xx} \mathcal{E}(t, x) = \partial_{xt} \mathcal{E}(t, x) = 0, \\ [\partial_{xxt} \mathcal{E}(t, x)]^2 = \partial_{xtt} \mathcal{E}(t, x) \cdot \partial_{xxx} \mathcal{E}(t, x)\} = \emptyset. \quad (20)$$

We show that in this case the set $I := I_1 \cup I_2$ only contains isolated points. Assume by contradiction that t is an accumulation point of I . Thus we must have a sequence $t_n \rightarrow t \in I_1$ with $t_n \in I_2$ for all $n \geq 1$. By Lemma 12 we have $\partial_{xt} \mathcal{E}(t_n, x(t_n)) = 0$ for all n . Since $\partial_{xx} \mathcal{E}(t_n, x(t_n)) = \partial_{xt} \mathcal{E}(t_n, x(t_n)) = 0$, taking the limit as $n \rightarrow \infty$ we get $\partial_{xx} \mathcal{E}(t, x(t)) = \partial_{xt} \mathcal{E}(t, x(t)) = 0$. Therefore, by the second statement of Lemma 14, the equation (13) has a unique solution $-\partial_{xxt} \mathcal{E}(t, x(t))/\partial_{xxx} \mathcal{E}(t, x(t))$. This implies that

$$[\partial_{xxt} \mathcal{E}(t, x)]^2 = \partial_{xtt} \mathcal{E}(t, x) \cdot \partial_{xxx} \mathcal{E}(t, x).$$

However, since $\partial_{xx} \mathcal{E}(t, x(t)) = \partial_{xt} \mathcal{E}(t, x(t)) = 0$ and $\partial_x \mathcal{E}(t, x(t)) \in \{-1, 1\}$ (by Lemma 8), we obtain a contradiction to the assumption (20). The proof is completed. \square

6. PROOF OF THEOREM 4

In this section, we prove Theorem 4.

Proof. Step 1. Since $x(\cdot)$ is a BV function, we have $L := \sup_{0 \leq t \leq T} |x(t)| < \infty$. For any $t \in [0, T]$, define

$$\mathcal{F}(t) := \{x \in [-L, L] : |\partial_x \mathcal{E}(t, x)| = 1\}.$$

We shall show that there exists $\varepsilon > 0$ independent of t such that if $x, y \in \mathcal{F}(t)$ and $x \neq y$, then $|x - y| \geq \varepsilon$.

Indeed, we assume by contradiction that there exists a sequence $\{t_n\}_{n=1}^\infty \subset [0, T]$ and $x_n, y_n \in \mathcal{F}(t_n)$ such that $x_n < y_n$ and $|x_n - y_n| \rightarrow 0$. By compactness, after passing to subsequences if necessary, we may assume that $t_n \rightarrow t_0$, $x_n \rightarrow x_0$ and $y_n \rightarrow x_0$. Using the continuity of $\partial_x \mathcal{E}$, we have $|\partial_x \mathcal{E}(t_0, x_0)| = 1$.

On the other hand, since $|\partial_x \mathcal{E}(t_n, x_n)|^2 = 1 = |\partial_x \mathcal{E}(t_n, y_n)|^2$, by applying Rolle's Theorem for the function $z \mapsto |\partial_x \mathcal{E}(t_n, z)|^2$, we can find an element $z_n \in (x_n, y_n)$ such that $\partial_{xx} \mathcal{E}(t_n, z_n) = 0$. Taking $n \rightarrow \infty$, we obtain $\partial_{xx} \mathcal{E}(t_0, x_0) = 0$.

Thus $|\partial_x \mathcal{E}(t_0, x_0)| = 1$ and $\partial_{xx} \mathcal{E}(t_0, x_0) = 0$, which contradicts to the assumption (H5). Therefore, there exists $\varepsilon > 0$ independent of t , such that $|x - y| \geq \varepsilon$ for all $x, y \in \mathcal{F}(t)$ and $x \neq y$.

Step 2. We assume that $x(\cdot)$ jumps at t , namely $x(t^-) \neq x(t^+)$, here

$$x(t^-) := \lim_{s \uparrow t} x(s) \text{ and } x(t^+) := \lim_{s \downarrow t} x(s).$$

We shall show that $|x(t^-) - x(t^+)| \geq \varepsilon$.

From the weak local stability of $x(\cdot)$, we have $|\partial_x \mathcal{E}(t, x(t^-))| \leq 1$ and $|\partial_x \mathcal{E}(t, x(t^+))| \leq 1$. If $|\partial_x \mathcal{E}(t, x(t^-))| = 1 = |\partial_x \mathcal{E}(t, x(t^+))|$, then by Step 1 we already get $|x(t^-) - x(t^+)| \geq \varepsilon$. Hence, let us assume that

$$\min\{|\partial_x \mathcal{E}(t, x(t^-))|, |\partial_x \mathcal{E}(t, x(t^+))|\} < 1. \quad (21)$$

Using the energy-dissipation upper bound, we get

$$|x(t^+) - x(t^-)| \leq \mathcal{E}(t, x(t^-)) - \mathcal{E}(t, x(t^+)) = \left| \int_{x(t^+)}^{x(t^-)} \partial_x \mathcal{E}(t, z) dz \right| \leq \int_I |\partial_x \mathcal{E}(t, z)| dz \quad (22)$$

where I is the closed interval between $x(t^-)$ and $x(t^+)$.

From (21) and (22), we conclude that there exists y between $x(t^-)$ and $x(t^+)$ such that $|\partial_x \mathcal{E}(t, y)| > 1$. Since $|\partial_x \mathcal{E}(t, x(t^-))| \leq 1 < |\partial_x \mathcal{E}(t, y)|$, there exists z_- between $x(t^-)$ and y such that $|\partial_x \mathcal{E}(t, z_-)| = 1$ (here z_- may be equal to $x(t^-)$). Similarly, there exists z_+ between $x(t^+)$ and y such that $|\partial_x \mathcal{E}(t, z_+)| = 1$ (here z_+ may be equal to $x(t^+)$). Since $z_+ \neq z_-$, we have $|z_+ - z_-| \geq \varepsilon$ by Step 1. Thus $|x(t^+) - x(t^-)| \geq |z_+ - z_-| \geq \varepsilon$.

Step 3. Thus by Step 2, any jump step is not less than ε . Since $x(\cdot)$ is a BV function, it can only have finitely many jumps. \square

7. APPENDIX: TECHNICAL PROOFS

7.1. Proof of Lemma 5. We start by an elementary result.

Lemma 17. *For any closed set C in \mathbb{R}^d , there exists a smooth function φ such that $\varphi : \mathbb{R}^d \rightarrow [0, 1]$ and $\varphi^{-1}(0) = C$.*

Proof. Since the set $\mathbb{R}^d \setminus C$ is open, we can find a family of open balls $\{B_n\}$ such that

$$\mathbb{R}^d \setminus C = \bigcup_{n \in \mathbb{N}} B_n.$$

Moreover, a classical result tells us that, for any $n \in \mathbb{N}$, there exist $\varphi_n : \mathbb{R}^d \rightarrow [0, 1]$ such that φ_n is of class C^∞ and $\varphi_n^{-1}(0) = \mathbb{R}^d \setminus B_n$.

Take $\varphi := \sum_{n \in \mathbb{N}} \alpha_n \varphi_n$ with $\alpha_n > 0$ for all n . This implies $\varphi^{-1}(0) = C$.

Now for every $n \in \mathbb{N}$, we choose α_n such that $\|D^k \varphi_n\|_\infty \cdot \alpha_n \leq 2^{-n}$ for all $k = 0, 1, \dots, n$. It is easy to check that $\varphi(\mathbb{R}^d) \in [0, 1]$ and φ is of class C^∞ . This completes the proof of Lemma 17. \square

Now we are ready to give the proof of Lemma 5.

Proof. Define

$$C_1 := \{(t, x) \mid x \geq u(t^-)\}, \quad C_2 := \{(t, x) \mid x \leq u(t^+)\}.$$

We show that C_1 and C_2 are closed sets in \mathbb{R}^2 . For example, to prove that C_1 is closed, we need to show that if a sequence $\{(t_n, x_n)\}_{n \geq 1} \subset C_1$ converges to (t_0, x_0) , then $(t_0, x_0) \in C_1$, namely $x_0 \geq u(t_0^-)$. Indeed, if $s < t_0$, then for n large enough we have $t_n > s$, and hence $x_n \geq u(t_n^-) \geq u(s)$. Thus $x_0 = \lim x_n \geq u(s)$ for all $s < t_0$, which implies that $x_0 \geq \lim_{s \uparrow t_0} u(s) = u(t_0^-)$. Thus C_1 is closed. Similarly, we have C_2 is closed.

Applying Lemma 17 we can choose two smooth functions $g_1 : \mathbb{R}^2 \rightarrow [0, 1]$ and $g_2 : \mathbb{R}^2 \rightarrow [0, 1]$ such that

$$g_1^{-1}(0) = C_1 \text{ and } g_2^{-1}(0) = C_2.$$

We define

$$g(t, x) := g_2(t, x) - g_1(t, x) \text{ for all } (t, x) \in [0, T] \times \mathbb{R}.$$

It is straight-forward to see that the function g has all desired properties. \square

7.2. Proof of Lemma 6.

Proof. Step 1. For any $h > 0$, we have

$$x(t+h) - x(t) = \mu((0, t+h]) - \mu((0, t]) = \mu((t, t+h])$$

Thus, we can rewrite

$$A = \left\{ t \in (0, T) \setminus J \mid \liminf_{h \rightarrow 0} \left| \frac{\mu((t, t+h])}{h} \right| < \infty \right\}.$$

If we define

$$A_k := \left\{ t \in (0, T) \setminus J \mid \left| \frac{\mu((t, t+h])}{h} \right| < k \text{ for some } h > 0 \text{ arbitrary close to } 0 \right\},$$

then $A \subset \bigcup_{k=1}^{\infty} A_k$. We can prove that A_k is Borel, and hence it is $|\mu_s|$ -measurable. We will obtain $|\mu_s|(A) = 0$ if we can check $|\mu_s|(A_k) = 0$ for all k .

Step 2. Since $\mu_s \perp \mathcal{L}^1$, there exists a Borel set S_p such that $|\mu_s|(S_p^c) = 0$ and $\mathcal{L}^1(S_p) = 0$. For any $\varepsilon > 0$ there exists an open set $U_\varepsilon \supset S_p$ such that $\mathcal{L}^1(U_\varepsilon) < \varepsilon$. We have

$$\mu_s(A_k) = \mu_s(A'_k),$$

where $A'_k = A_k \cap U_\varepsilon$.

Step 3. Next, we consider the following family \mathcal{F} covering A'_k where

$$\mathcal{F} = \{[t, t+h] : t \in A'_k \text{ and } h \text{ is chosen such that } [t, t+h] \subset U_\varepsilon \text{ and } |\mu|((t, t+h]) < kh\}.$$

Notice that since t does not belong to J , $\mu([t, t+h]) = \mu((t, t+h])$. We refine \mathcal{F} to \mathcal{F}' such that \mathcal{F}' still covers A'_k as follows: if $I \in \mathcal{F}$ is a subset of another interval of \mathcal{F} , or I is a subset of the union of two other intervals of \mathcal{F} , then we omit I .

After refining \mathcal{F} , we obtain the family \mathcal{F}' with the following properties.

- P1. No interval of \mathcal{F}' is a subset of another interval of \mathcal{F}' .
- P2. Three different intervals of \mathcal{F}' always have no common element (otherwise, two of them cover the remaining one). As a consequence, any $t \in A'_k$ is covered by at most 2 intervals of \mathcal{F}' .
- P3. Any interval of \mathcal{F}' is disjoint from all of the others except at most 2 intervals. In fact, if $J \cap I \neq \emptyset$, $J \not\subset I$ and $I \not\subset J$, then J must contain precisely one of the two boundary points of I . Therefore, by property P2, there are at most 2 intervals of \mathcal{F}' , which are different from I and have nontrivial intersections with I .
- P4. For any $I \in \mathcal{F}'$, there is a nontrivial interval $(a, b) \subset I$ such that

$$(a, b) \cap \left(\bigcup_{I' \in \mathcal{F}' \setminus \{I\}} I' \right) = \emptyset.$$

In fact, assume that for all $(a, b) \subset I$, there exists $J \in \mathcal{F}'$, $J \neq I$, such that $(a, b) \cap J \neq \emptyset$. By property P3, there are (at most) two intervals $J_1, J_2 \in \mathcal{F}'$ such that for all $(a, b) \subset I$, then either $(a, b) \cap J_1 \neq \emptyset$ or $(a, b) \cap J_2 \neq \emptyset$. It implies that $I \subset J_1 \cup J_2$, which is a contradiction.

- P5. \mathcal{F}' is at most countable. In fact, by property P4, each $I \in \mathcal{F}'$ contains a nontrivial interval (a, b) which has empty intersection with all of the other intervals of \mathcal{F}' . In this interval (a, b) we can choose a rational number c_I . Since $I \mapsto c_I$ is injective from \mathcal{F}' into \mathbb{Q} , we conclude that \mathcal{F}' is at most countable.

P6. We can divide \mathcal{F}' into three subfamilies $\mathcal{F}'_1, \mathcal{F}'_2, \mathcal{F}'_3$, such that each subfamily is disjoint. This can be done by induction using property P2.

Step 4. Now we have

$$|\mu_s|(A_k) = |\mu_s|(A'_k) \leq |\mu_s|\left(\bigcup_{I \in \mathcal{F}'} I\right) \leq \sum_{I \in \mathcal{F}'} |\mu_s|(I) \leq \sum_{j=1}^3 \left(\sum_{I \in \mathcal{F}'_j} |\mu_s|(I) \right).$$

Recall that any $I \in \mathcal{F}'$ satisfies $|\mu_s|(I) \leq k\mathcal{L}^1(I)$. Moreover, for any $j = 1, 2, 3$, the family \mathcal{F}'_j contains disjoint intervals $I \subset U_\varepsilon$. Therefore,

$$\sum_{I \in \mathcal{F}'_j} |\mu_s|(I) \leq k \sum_{I \in \mathcal{F}'_j} \mathcal{L}^1(I) = k\mathcal{L}^1(U_\varepsilon) \leq k\varepsilon \text{ for all } j = 1, 2, 3.$$

Thus

$$|\mu_s|(A_k) \leq 3k\varepsilon.$$

Because it holds true for any $\varepsilon > 0$, we conclude that $|\mu_s|(A_k) = 0$. This completes the proof of Lemma 6. \square

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